where

$$\begin{array}{lll} p=0, & B=4-b & \text{for } b_{2k}^c \\ p=0, & B=4-b+2q^2/b & \text{for } b_{2k}^s \\ p=1, & B=1+q-b & \text{for } b_{2k+1}^c \\ p=1, & B=1-q-b & \text{for } b_{2k+1}^s \end{array} \quad (9)$$

The combination of (7) and (8) can provide exact characteristic values of the modified Mathieu functions with large range of m and q.

C. Numerical Results

As a check of this method, we calculated 200 successive modes for elliptical waveguides with different ellipticities. Table. I lists the lowest 100 successive modes with ellipticities $e = 0.1$, 0.5 and 0.9. It is obvious from Table. I that the eigenmode sequence is a function of ellipticity, i.e., elliptical waveguide with different ellipticity has different eigenmode sequence. However, the main mode of the waveguide is always TE_{c11}. The first high order mode is TE_{s11} when $e < 0.8546001$ while it becomes TE_{c21} when $e > 0.8546001$.

As a large number of numerical calculation are required to determine the cutoff wavelength for a given mode and ellipticity, we presented here the curvefitting expressions for the determination of the cutoff wavelength of the lowest 10 order modes. The formulas for the different modes and their ranges of validity are given in Table II. Compared with previous works [5], [7], the expressions presented here have higher accuracy and are valid for wider range of ellipticity.

IV. Conclusion

We can conclude from above discussion that: 1) the modified continued fractional method suggested in this paper is suitable to calculate the characteristic values of the modified Mathieu functions with arbitrary order m and value q. 2) directly calculating the parametric zeros of the modified Mathieu functions of the first kind and their derivatives is an effective and easy way to determine the cutoff wavelength for a given elliptical waveguide, and ensures no omission of high order modes in the eigenmode sequence. 3) The normalized cutoff wavelength for the lowest 100 successive modes are presented, and the curvefitting expressions for the determination of the cutoff wavelength of the lowest 10 order modes are given, which have higher accuracy than previous calculations and are valid for wider range of ellipticity.

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A New Electric Field Integral Equation for Heterogeneous Dielectric Bodies of Revolution

Mark S. Viola

***Abstract*—In this paper, a novel electric field integral equation (EFIE) is developed for rotationally-symmetric heterogeneous dielectric bodies. This EFIE has several attractive features. Firstly, the azimuthal field component has been eliminated in this formulation thereby reducing the number of scalar unknowns from three to two. Secondly, it is a pure-integral equation in which there are no terms involving derivatives of the field components. Finally, this description is devoid of any highly singular kernel which would require a principal-value evaluation of the associated integral. These attributes render this formulation advantageous for both computational and theoretical pursuits.**

I. Introduction

Rigorous analysis of electromagnetic phenomena within heterogeneous dielectric regions commonly proceeds from an integral or integro-differential equation for the electric field [1]–[5]. Construction of such an EFIE relies upon the identification of an equivalent volume density of polarization current. Inherently, this formulation is a volume integral equation having three scalar unknowns. Thus, its solution potentially poses a computationally intensive problem. Additional complications arise when the EFIE is cast in the form involving the electric dyadic Green's function [6]–[9]. However, the presence of certain symmetries allows the formulation of an alternative integral equation that provides both computational and theoretical advantages.

In this paper, a novel electric field integral equation (EFIE) is developed for heterogeneous dielectric bodies of revolution. It is assumed that the permittivity profile is azimuthally invariant. By exploiting the prevailing symmetry, straightforward analysis yields an EFIE having several appealing attributes. Firstly, the azimuthal field component is eliminated from the formulation in favor of the remaining (transverse) components. This reduction in the number of scalar unknowns from three to two facilitates numerical solution via standard techniques (e.g., the method of moments). Secondly, it is a rigorous pure integral equation for the transverse field components as opposed to an integro-differential one; no terms involving derivatives of the field components appear. Finally, the singularities of the kernels within this formulation are sufficiently weak, avoiding the necessitation of a principal-value integral and the corresponding depolarizing dyadic [7].

Throughout this paper, it shall be assumed that all media are linear, isotropic and magnetically homogeneous. Furthermore, the time dependence is harmonic ($e^{j\omega t}$) and is suppressed.

II. Volume-Surface Integral Equation Description

Attention is focused on Fig. 1, which depicts a dielectric body of revolution immersed in a uniform surround. A coordinate system is established such that the z-axis coincides with the axis of revolution. Open domain V, having boundary surface S with outer unit normal $\hat{n}$, is the region for the dielectric and is electrically characterized through its permittivity profile $\epsilon(\vec{r})$. In order to provide a well-posed problem, it is assumed that the closed region $\overline{V}$ is regular and that ϵ is

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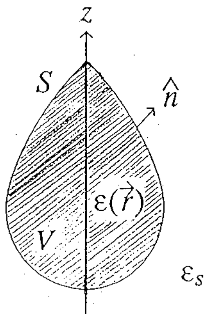

Fig. 1. Geometry for a heterogeneous dielectric body of revolution.

continuously differentiable in V (see [1]). Additionally, it is assumed that $\epsilon(\vec r)$ is axially symmetric. The background is a homogeneous surround and is characterized by permittivity ϵ_s. System excitation is provided by an impressed field $\vec E^i$ maintained by a source system external to V.

Quantification of the electromagnetics for the situation described above is provided by the EFIE

$$\vec E(\vec r)=\vec E^i(\vec r)+(k_s^2+\nabla\nabla\cdot)\vec\Pi^e(\vec r), \vec r\in V \tag{1}$$

where the electric Hertzian potential $\vec\Pi^e$ is given by

$$\vec\Pi^e(\vec r)=\int_V \frac{\delta\epsilon(\vec r')}{\epsilon_s}G(\vec r|\vec r')\vec E(\vec r')\,dV'. \tag{2}$$

Here, $\delta\epsilon=\epsilon-\epsilon_s$ is the contrast between permittivity within volume V and that of its uniform surround, k_s is the wavenumber within the unbounded background, and G is the Green's function for the unbounded surround

$$G(\vec r\,|\,\vec r')=\frac{e^{-jk_sR}}{4\pi R} \tag{3}$$

where $R=\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2}$. The integro-differential equation as given by (1) and (2) may be converted into a pure integral equation through a rigorous interchange of integral and differential operations [6]. That converted form contains a highly singular kernel such that evaluation of the integral requires specification of an excluding region along with the corresponding depolarizing dyadic [7]. In practice, the difficulty of managing the singularity is usually circumvented by using one of the following approaches: 1) a suitable spectral representation of the dyadic Green's function is used and followed by an interchange of spatial and spectral operators, 2) application of a smooth testing operator in the implementation of the method of moments naturally reduces the strength of the singularity, and 3) the EFIE formulation is recast into a volume-surface integral equation (VSIE). The latter approach is taken here to facilitate the development of a new EFIE for the field components transverse to ϕ.

Conversion of (1) into a volume-surface EFIE is readily accomplished by using the relationship $\nabla G=-\nabla' G$, the product rule $\nabla\cdot(\phi\vec A)=\phi\nabla\cdot\vec A+\nabla\phi\cdot\vec A$, and the divergence theorem [10]. Then it is easy to show that

$$\nabla\cdot\vec\Pi^e(\vec r)=-\oint_S\frac{\delta\epsilon(\vec r')}{\epsilon_s}\vec E(\vec r')\cdot\hat n' G(\vec r|\vec r')\,dS' + \int_V\frac{1}{\epsilon_s}\nabla'\cdot(\delta\epsilon(\vec r')\vec E(\vec r'))G(\vec r|\vec r')\,dV' \tag{4}$$

where the legitimacy of differentiating under the integral was verified in [6]. Application of Gauss's Law $\nabla'\cdot(\epsilon(\vec r')\vec E(\vec r'))\equiv 0$ reveals

$$\nabla'\cdot(\delta\epsilon(\vec r')\vec E(\vec r'))=\nabla'\cdot(\epsilon(\vec r')\vec E(\vec r'))-\epsilon_s\nabla'\cdot\vec E(\vec r') =\epsilon_s\frac{\nabla'\epsilon(\vec r')}{\epsilon(\vec r')}\cdot\vec E(\vec r') \tag{5}$$

whereby substitution of (2), (4), and (5) into (1) yields the VSIE

$$\vec E(\vec r)=\vec E^i(\vec r)+\int_V\left\{k_s^2\frac{\delta\epsilon(\vec r')}{\epsilon_s}\vec E(\vec r')G(\vec r|\vec r') + \frac{\nabla'\epsilon(\vec r')}{\epsilon(\vec r')}\cdot\vec E(\vec r')\nabla G(\vec r|\vec r')\right\}dV' - \oint_S\frac{\delta\epsilon(\vec r')}{\epsilon_s}(\vec E(\vec r')\cdot\hat n')\nabla G(\vec r|\vec r')\,dS' \tag{6}$$

for all $\vec r\in V$. Again, the interchange of differential and integral operators used to obtain (6) is rigorously justified by [6]. Note that for $\vec r$ in the open domain V, all the integrals in (6) converge without the necessity of a principal value evaluation. Although the surface integral contains the term ∇G, its limit as $\vec r\to\vec r'\in S$ exists [11]–[13]. Therefore, $\vec E$ may be continuously extended from open domain V to the closed domain $\overline V$ by defining $\vec E$ on the boundary as $\vec E(\vec r_s\in S)\equiv\lim_{\vec r\to\vec r_s}\vec E(\vec r)$ where the limit is approached from within V. Let $\overline{\overline I}$ designate the unit dyadic. Then, for all $\vec r\in S$,

$$\left[\overline{\overline I}+\frac{\delta\epsilon(\vec r)}{2\epsilon_s}\hat n\hat n\right]\cdot\vec E(\vec r) =\vec E^i(\vec r)+\int_V\left\{k_s^2\frac{\delta\epsilon(\vec r')}{\epsilon_s}\vec E(\vec r')G(\vec r|\vec r') + \frac{\nabla'\epsilon(\vec r')}{\epsilon(\vec r')}\cdot\vec E(\vec r')\nabla G(\vec r|\vec r')\right\}dV' - \oint_S\frac{\delta\epsilon(\vec r')}{\epsilon_s}(\vec E(\vec r')\cdot\hat n')\nabla G(\vec r|\vec r')\,dS'$$

where the surface integral exists as a Cauchy principal value [12]. However, the normal component of this surface integral is independent of the shape of the region which excludes the singularity [11]–[13]. Hence, the normal component of electric field, which is implicated in (6), can be extended continuously from V to $\overline V$ without need of a principal-value integral.

At this point, it should be emphasized that (6) is a general expression which is valid for dielectric bodies of arbitrary shape and permittivity. In the next section, specialization of (6) to axially-symmetric bodies is considered.

III. Derivation of a New VSIE for the Transverse Components

By exploiting the rotational symmetry of the permittivity ϵ, it is possible to formulate an EFIE that uncouples the azimuthal ($\hat\phi$) component from the transverse ones. Following a development analogous to that found in [14] for longitudinally-uniform dielectric waveguides, the electric field is decomposed as $\vec E\equiv\vec E_T+\hat\phi E_\phi$. Now by observing that the following relations hold for bodies of revolution

$$\hat n\cdot\vec E=\hat n\cdot\vec E_T$$
$$\nabla\epsilon\cdot\vec E=\nabla_T\epsilon\cdot\vec E_T$$

where $\nabla_T=\nabla-\hat\phi(1/\rho)(\partial/\partial\phi)$ is the transverse gradient operator, it is seen that the azimuthal part of (6) may be written

$$E_\phi(\vec r)=E_\phi^i(\vec r)+\int_V\left\{k_s^2\frac{\delta\epsilon(\vec r')}{\epsilon_s}(\hat\phi\cdot\vec E_T(\vec r'))G(\vec r|\vec r') + \frac{\nabla_T'\epsilon(\vec r')}{\epsilon(\vec r')}\cdot\vec E_T(\vec r')\frac{1}{\rho}\frac{\partial G(\vec r|\vec r')}{\partial\rho}\right\}dV' - \oint_S\frac{\delta\epsilon(\vec r')}{\epsilon_s}(\vec E_T(\vec r')\cdot\hat n')\nabla_T G(\vec r|\vec r')\,dS' + \int_V k_s^2\frac{\delta\epsilon(\vec r')}{\epsilon_s}E_\phi(\vec r')\cos(\phi-\phi')G(\vec r|\vec r')\,dV' \tag{7a}$$

while the transverse part of (6) becomes

$$\vec{E}_T(\vec{r}) = \vec{E}_T^i(\vec{r}) + \int_V \left\{ k_s^2 \frac{\delta\epsilon(\vec{r}')}{\epsilon_s}\{\bar{\bar{I}} - \hat{\phi}\hat{\phi}\} \cdot \vec{E}_T(\vec{r}')G(\vec{r}|\vec{r}') + \frac{\nabla_T'\epsilon(\vec{r}')}{\epsilon(\vec{r}')} \cdot \vec{E}_T(\vec{r}')\nabla_T G(\vec{r}|\vec{r}') \right\} dV' - \oint_S \frac{\delta\epsilon(\vec{r}')}{\epsilon_s}(\vec{E}_T(\vec{r}') \cdot \hat{n}')\nabla_T G(\vec{r}|\vec{r}')\, dS' + \hat{\rho} k_s^2 \int_V \frac{\delta\epsilon(\vec{r}')}{\epsilon_s} E_\phi(\vec{r}') \sin(\phi - \phi')G(\vec{r}|\vec{r}')\, dV'. \tag{7b}$$

Here $\vec{E}_T^i = \vec{E}^i - \hat{\phi}E_\phi^i$ is the transverse component of the impressed electric field and $\hat{\rho} = \hat{\phi} \times \hat{z}$.

Observe that if $\vec{E}_T$ is found, then it could be substituted back into (7a) thereby yielding an integral equation for E_ϕ alone. Yet, there is apparent coupling between E_ϕ and $\vec{E}_T$ through the last term in (7b). Elimination of E_ϕ in that term is possible by appealing to Gauss's law. From $\nabla' \cdot (\epsilon(\vec{r}')\vec{E}(\vec{r}')) \equiv 0$, the desired relationship is

$$\frac{1}{\rho'}\frac{\partial E_\phi(\vec{r}')}{\partial \phi'} = -\frac{\nabla' \cdot \epsilon(\vec{r}')\vec{E}_T(\vec{r}')}{\epsilon(\vec{r}')}. \tag{8}$$

At first, it seems doubtful that (8) may be of any use since it is E_ϕ and not its derivative with respect to ϕ that appears in (7b). However, by invoking the azimuthal invariance of the permittivity ϵ, an integration by parts scheme may be used to alter (7b) into a form that contains the left side of (8). As a preliminary step to implement that scheme, let a function $G_s(r|r')$ be defined such that

$$\frac{1}{\rho'}\frac{\partial G_s(\vec{r}|\vec{r}')}{\partial \phi'} = -\sin(\phi - \phi')G(\vec{r}|\vec{r}').$$

Then, using (3), it is easy to see that

$$G_s(r|r') = -\rho' \int G(\vec{r}|\vec{r}')\sin(\phi - \phi')\, d\phi' = \frac{1}{4\pi j k_s \rho}\{e^{-jk_s R} - e^{-jk_s R_0}\}$$

where $R_0 = \sqrt{\rho'^2 + (z - z)^2}$. Observe that the term involving $e^{-jk_s R_0}$ is simply an integration constant that has been chosen so that G_s is defined along the polar axis and satisfies the radiation condition.

Now applying integration by parts with respect to ϕ' to the last term in (7b) yields

$$\int_V \frac{\delta\epsilon(\vec{r}')}{\epsilon_s} E_\phi(\vec{r}') \sin(\phi - \phi')G(\vec{r}|\vec{r}')\, dV' = -\int_V \frac{\delta\epsilon(\vec{r}')}{\epsilon_s} G_s(\vec{r}|\vec{r}')\frac{1}{\rho'}\frac{\partial E_\phi(\vec{r}')}{\partial\phi'}\, dV' \tag{9}$$

where use was made of the periodic and single-valued nature of both E_ϕ and G_s. Using (8) along with the relation

$$\epsilon(\vec{r}')\nabla'\left(\frac{\delta\epsilon(\vec{r}')}{\epsilon_s\epsilon(\vec{r}')}\right) = \frac{\nabla'\epsilon(\vec{r}')}{\epsilon(\vec{r}')}$$

it is found that, after invoking a three-dimensional integration by parts, (9) becomes

$$\int_V \frac{\delta\epsilon(\vec{r}')}{\epsilon_s} E_\phi(\vec{r}') \sin(\phi - \phi')G(\vec{r}|\vec{r}')\, dV' = \oint_S \frac{\delta\epsilon(\vec{r}')}{\epsilon_s}(\vec{E}_T(\vec{r}') \cdot \hat{n}')G_s(\vec{r}|\vec{r}')\, dS' - \int_V \frac{\nabla_T'\epsilon(\vec{r}')}{\epsilon(\vec{r}')} \cdot \vec{E}_T(\vec{r}')G_s(\vec{r}|\vec{r}')\, dV' - \int_V \frac{\delta\epsilon(\vec{r}')}{\epsilon_s}\vec{E}_T(\vec{r}') \cdot \nabla_T' G_s(\vec{r}|\vec{r}')\, dV'. \tag{10}$$

Finally, substituting (10) into (7b) and collecting terms leads to the EFIE

$$\vec{E}_T(\vec{r}) = \vec{E}_T^i(\vec{r}) + \int_V k_s^2 \frac{\delta\epsilon(\vec{r}')}{\epsilon_s}\left\{[\bar{\bar{I}} - \hat{\phi}\hat{\phi}]G(\vec{r}|\vec{r}') + \hat{\rho}\nabla_T' G_s(\vec{r}|\vec{r}')\right\} \cdot \vec{E}_T(\vec{r}')\, dV' + \int_V \frac{\nabla_T'\epsilon(\vec{r}')}{\epsilon(\vec{r}')} \cdot \vec{E}_T(\vec{r}')[\nabla_T G(\vec{r}|\vec{r}') + \hat{\rho}k_s^2 G_s(\vec{r}|\vec{r}')]\, dV' - \oint_S \frac{\delta\epsilon(\vec{r}')}{\epsilon_s}(\vec{E}_T(\vec{r}') \cdot \hat{n}') \cdot [\nabla_T G(\vec{r}|\vec{r}) + \hat{\rho}k_s^2 G_s(\vec{r}|\vec{r}')]\, dS'. \tag{11}$$

for all $\vec{r} \in V$.

It is believed that (11) is a new formulation. Upon examination of the kernels appearing in this EFIE, it is seen that the highest order singularity is contributed by $\nabla_T G$. Terms involving G_s and its derivatives are continuous throughout the open region V. Hence, all of the integrals in (11) are well defined and are independent of the shape of the excluding region. The electric field may be extended as a continuous function throughout the closed domain $\bar{V}$ by the limiting procedure previously discussed. Aside from this, it is perhaps of more practical importance that a great computational advantage can be gained by using (11) due to the reduction in the number of unknowns. Solution to (11) may be substituted into (7a) resulting in an EFIE for E_ϕ. Once (7a) is solved, the equivalent volume density of polarization is known within V and the scattered field external to V may then be computed using (1) and (2).

IV. Conclusions

An electric field integral equation for the field components transverse to $\hat{\phi}$ has been derived for rotationally-symmetric heterogeneous bodies. It is believed that the new formulation should be an asset for both theoretical and computational endeavors. It enhances the efficiency of numerical computations by yielding a formulation for which the number of unknowns is reduced from three to two. Theoretically, it affords the luxury of providing a rigorous pure-integral equation description for which the singularities of the associated kernels are weak. (Although the surface integral in (11) contains the term $\nabla_T G$, its limit as $\vec{r} \to \vec{r}' \in S$ exists.) This avoids the necessitation of both a principal-value integral whose value depends upon the shape of the excluding region and the corresponding depolarizing dyadic.

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Computation of Proper and Improper Modes in Multilayered Bianisotropic Waveguides

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Abstract—**An efficient numerical method is presented to determine the loci of both the proper and complex improper modes of a multilayered bianisotropic planar waveguide. The propagation constants of the waveguide modes are expressed in terms of the zeros of a specific analytic function. The use of appropriate integration zero-searching methods is proposed since information about the possible number of complex improper modes cannot be previously extracted. The general formulation presented here has been applied to the study of the complex improper modes of single and two-layer structures containing magnetized ferrites. It has been found that the transition from physical proper to complex improper modes is made throughout a nonphysical real improper mode.**

I. Introduction

The grounded multilayered planar waveguide is the basic background of microstrip antennas, microstrip patch resonators and open dielectric waveguides for integrated optics and millimeter-wave integrated circuits [1], [2]. A topic which demands recent and increasing interest is the effect of an increasing number of layers [3] and substrate complexity [4] on the radiation pattern in antennas, the resonant frequency of patch resonators and the propagation characteristics in open dielectric waveguides. Computation and further analysis of the Green's function of the involved configuration can become essential. This analysis is usually carried out by studying the singularities of the Green's function: the branch-point singularities account for the free dipole radiation and the pole singularities for the background radiation and guided modes [1]. Thus, finding the pole singularities, which are located on a two-sheeted Riemann surface, is a preliminary step in obtaining closed-form representations of the Asymptotic Green's Function (AGF) [3], [5]. Assuming that the upper sheet of the Riemman surface is defined as fulfilling the radiation condition, the poles located on this sheet (*proper* sheet) form a finite and real subset which corresponds to the *bounded* modes guided by the layered slab. On the other hand, the complex and infinite subset of poles located on the bottom (*improper*) sheet, correspond to unbounded modes which are usually called *leaky* modes [1].

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There are some works in the literature devoted to the computation (and further application) of the proper and real improper modes [6]. Complex improper modes are treated in [5], where the possible significance of these complex modes is also discussed. Nevertheless, to our knowledge the substrate of the considered structures was assumed to be isotropic. Thus, the purpose of the present paper is to provide an efficient numerical method to determine the location of proper and improper waveguide modes in a planar waveguide with layered bianisotropic substrate. The method is based on the computation of the zeros of a specific *analytic* complex function (with no poles or branch-cut singularities). The search for the zeros of this function is carried out using an integral scheme which enables analysis of the complex plane (determining the number of zeros included within the closed integration contour) and accurate computation of the zeros.

II. Analysis

In this section, the dispersion relation of a bianisotropic layered waveguide will be obtained. Note that this waveguide ranges from a simple grounded/covered/slab dielectric waveguide to waveguides with gyrotropic (semiconductor and/or ferrites biased by an arbitrarily oriented external d.c. magnetic field) and/or chiral layered substrate. The theory presented here is also applicable to those multilayered planar waveguides whose upper and bottom boundary conditions can be expressed as impedance or admittance dyads.

This work pointedly formulates the dispersion relation of the generic waveguide under consideration in terms of the zeros of an *analytic* function. This fact will be relevant in connection with the zero-searching procedure since the usual methods work efficiently when applied to analytic functions in the search region. The following dependence of the electromagnetic field at the plane of the interfaces (that is, the x, z components) $\mathbf{X} = (E_x, E_z, H_x, H_z)$, is assumed: $\mathbf{X}(x,y,z,t) = \exp(-j\omega t)\exp(-j\mathbf{k}_t\cdot\boldsymbol{\rho})\mathbf{X}(y)$, where ω is the angular frequency, $\boldsymbol{\rho} = x\mathbf{a}_x + z\mathbf{a}_z$ and $\mathbf{k}_t = k_x\mathbf{a}_x + k_z\mathbf{a}_z$ is the wavevector. As is shown in [7], the $\mathbf{X}$ vector inside each layer (denoted by the subscript i) is given in terms of a certain exponential matrix and a certain reference value, that is: $\mathbf{X}_i(y) = \exp(j\omega[\mathbf{Q}]_i y)\cdot\mathbf{X}_i(0)$. The explicit form of each element of the (4×4) $j\omega[\mathbf{Q}]_i$ matrix as a function of the layer characteristics is shown in [8].

Once fields at the upper interface of each layer are expressed in terms of fields at the bottom interface of each layer, we can express the field at the upper interface of the whole waveguide, $\mathbf{X}^u$, in terms of the field at the bottom interface of the waveguide, $\mathbf{X}^b$, by applying the continuity condition of the $\mathbf{X}$ vector at each intermediate interface. Thus, assuming that N_l is the total number of layers, the following matrix relation is obtained: $\mathbf{X}^u = [\mathbf{A}]\cdot\mathbf{X}^b$, where the $[\mathbf{A}]$ matrix is given by $[\mathbf{A}] = \prod_{i=1}^{N_l}\exp(j\omega[\mathbf{Q}]_i h_i)$ — h_i is the height of the i-th layer —.

The above matrix relation, together with the matrix impedance relations of $\mathbf{E}_t = (E_x, E_z)$ and $\mathbf{H}_t = (H_x, H_z)$ at the upper and bottom interfaces of the waveguide, enables writing the following matrix equations in terms of the $(2\times 2)[\mathbf{A}_{ij}]$ submatrices of the $[\mathbf{A}]$ matrix and the impedance matrices, $[\mathbf{Z}_u]$ and $[\mathbf{Z}_b]$:

$$\mathbf{E}_t^u = [\mathbf{A}_{11}]\cdot\mathbf{E}_t^b + [\mathbf{A}_{12}]\cdot\mathbf{H}_t^b \quad (1)$$
$$\mathbf{H}_t^u = [\mathbf{A}_{21}]\cdot\mathbf{E}_t^b + [\mathbf{A}_{22}]\cdot\mathbf{H}_t^b \quad (2)$$
$$\mathbf{E}_t^u = [\mathbf{Z}_u]\cdot\mathbf{H}_t^u \quad (3)$$
$$\mathbf{E}_t^b = [\mathbf{Z}_b]\cdot\mathbf{H}_t^b. \quad (4)$$